Journal of Sound and Vibration (1997) **201**(5), 535–554

ENERGY MODELS OF ONE-DIMENSIONAL, MULTI-PROPAGATIVE SYSTEMS

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(*Received* 24 *June* 1995, *and in final form* 26 *September* 1996)

For a number of years, a model well suited to medium and high frequencies in structures, and called Energy Flow analysis, has been studied in order to generalize Statistical Energy Analysis. This model is based on a thermal analogy: a law analogous to Fourier's law for heat flow is involved. This relationship, which relates the energy flow to the energy density, leads to a differential equation similar to the heat conduction equation in steady state conditions. The aim of this study is to generalize previous works on one-dimensional structures. A wave approach is adopted. It is shown that Fourier's law is valid for one symmetric propagation mode (one group velocity). However this law has to be modified for non-symmetric propagation modes or multi-mode propagation. In each case, the wave approach determines the relationship betwen energy density and energy flow. Finally, the theoretical models are illustrated with several examples of waveguides: an Euler–Bernoulli beam on an elastic support, pipes carrying moving fluid and a Timoshenko beam.

1. INTRODUCTION

An important current challenge in engineering design procedure is to reduce vibration and noise, especially to satisfy increasingly stringent government regulations in respect to urban and factory machinery noise and vibration. Vibration and noise predictive tools are thus required to facilitate design of quiet and robust machines, by defining structural vibration and noise paths. This allows a rigorous and efficient passive and/or active control strategy to be correctly implemented.

Unfortunately, there is no simple efficient predictive tool that is able to represent correctly throughout the entire audible frequency range, structural borne vibration and noise paths of systems with structural complexity. In fact, the finite element method and the boundary element method, which are the most widely used industrial tools, are effective only for a relatively low frequency range. The main weakness in mid- and high-frequency ranges of those methods, according to the authors, is its inability to deal correctly with damping phenomena. Indeed, such methods correctly predict a well separated resonant behaviour (model frequency range) but fail for dynamics with a high modal overlap (non-modal frequency range). As this last property is the main feature of the mid- and high-frequency dynamics, those methods cannot be used with confidence and with reasonable computational cost to analyze structural vibration and noise in this domain.

In this context, statistical energy analysis SEA [1] appeared to be an attractive alternative to the classical predictive tools. From a discrete analysis of the energy exchanges in complex built-up systems, this method permits a space and frequency determination of quadratic vibro-acoustical response. The weakness of the SEA stems from its heuristic and constrained hypothesis: That is, only an energy level for each subsystem is estimated with

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SEA. A large number of publications have been dedicated to SEA industrial validations, to uncertainties or to SEA extensions. From an SEA uncertainties study [2], it appears that the statistical analysis provided is well suited to systems with a large modal density and modal overlap, or for very high frequencies [2, 3]. It should be noted that the basic premise of SEA is that the energetic exchange can be modelled in a way which is similar to heat diffusion and involves an energy transfer from the hotter subsystem to the cooler one.

Among possible extensions of statistical energy analysis, is asymptotic modal analysis (AMA) introduced by Dowell and Kubota [4], which pointed out and demonstrated the majority of the SEA features from a pure modal description. AMA has been applied theoretically and experimentally with some success to many vibrational and acoustical configurations [5].

In contrast, some energy models are based on a pure propagative description. A random wave field is introduced and the propagative waves are assumed to be uncorrelated. This hypothesis is a most stringent characteristic of Wave Intensity Analysis (WIA) [6]. An hypothesis such as this is also used in the ray structural method proposed by Bondoux, Aquilina and Parot [7] for the study of beam networks up to high frequencies.

As another alternative, a simplified energy model has been propsed in order to improve SEA results. The original work of Belov, Rybak and Tartakovski [8] ranks among the earliest developments of such power flow models. The authors analyzed the propagation of flexural and longitudinal vibrational energy in absorbing structure by means of an energetic approximation. They derived a differential equation of heat-conduction type to characterize the spread of energy throughout the structure. Buvailo and Ionov [9] proposed a finite element method investigation of such an energy model to obtain some vibro-acoustical characteristics of structures at high audio frequencies. Nefske and Sung [10] applied the thermal analogy to deal with the power flow in a straight transversely vibrating beam, and also developed a power flow finite element model (PFFEM) to solve numerically the resulting energy equation. Wohlever, Bouthier and Bernhard [11, 12] have given further results concerning the energy model of rods, Euler–Bernoulli beams, membranes and plates. Lase, Le Bot, Ichchou and Jezequel [13–17] developed the so-called general energy method (GEM), which gives a complete energy description of beams and rods, from the use of active energy flow and total energy density to represent the pure propagative behaviour and the use of reactive energy flow and Lagrangian energy density for the modal aspect. The simplified energy model (SEM) is provided as a simplification of GEM by the elimination of the modal characteristics.

From this short description of the state of the art, it is obvious that physical systems involving multi-propagative modes have never been considered before. For instance, concerning the one-dimensional configuration, only rods and Euler–Bernoulli beams have been treated in the mid- and high frequency literature. However, from the author's point of view, a simple rod or Euler–Bernoulli beam model does not represent high frequency dynamics correctly. At high frequencies, the dynamical and acoustical behaviour of a beam is mainly influenced by the deformations within its section, which are not taken into account in the Euler–Bernoulli model. Thus, the propagative behaviour of a system becomes richer as the frequency increases. In fact, this study can be considered to be the first step in the thin walled energy description of waveguides.

Three classes of waveguides are studied here [18]. The first class is that of one-dimensional systems having one symmetric propagative mode (only one group velocity). It is demonstrated that the flow of energy within such systems can be modelled correctly by using the heat conduction analogy. The simplified energy model obtained is then applied to the case of a simple beam on an elastic support. The second class studied is that of systems with one non-symmetric propagative mode. It is shown that the energy

flow cannot be represented by a thermal field model. The simplified energy model then obtained involves a corrective term. As an illustration, the example of pipes conveying fluid is presented. Finally, the last class is that of systems with two symmetric propagative modes, and a Timoshenko beam model is developed.

2. PREAMBLE FOR THE SIMPLIFIED ENERGY METHOD DERIVATION

As mentioned and discussed in reference [18], the set of assumptions required to derive the energy models can be summarized as follows: (i) linear and elastic systems; (ii) steady state conditions with harmonic excitation of frequency ω; (iii) slight hysteretic damping loss factor ($\eta \ll 1$); (iv) far from singularities, evanescent waves are neglected; (v) interference among propagative waves is not taken into account.

In what follows, the Young's modulus of the elastic structures is denoted as E_0. A complex modulus E is defined as $E \stackrel{\text{def}}{=} E_0(1 + i\eta)$ (i.e., the hysteretic damping model is used).

Two energy variables are involved: the energy density W, defined as the sum of the kinetic energy density and the potential energy density, and the active energy flow P, defined as the real part of the complex energy flow.

The first step in the development of the energy models is the well known local energy balance, written in a non-loaded region, as shown in Figure 1, as

$$\mathrm{d}P/\mathrm{d}x + p_{diss} = 0. \tag{1}$$

The choice of hysteretic damping implies a proportionality between the poser density being dissipated and the potential energy density. As in Statistical Energy Analysis, no distinction is made between the kinetic energy density and the potential energy density (i.e., only the sum of the two is considered). The power density being dissipated is then

$$p_{diss} = \eta\omega W. \tag{2}$$

It should be noted that such a proportionality between the power density being dissipated and the energy density can be extended to further kinds of damping models. In particular, according to assumptions (iv) and (v) it has been established [13] that the hysteretic damping is equivalent to the viscous one, as the kinetic energy density is equal to the potential energy density.

In order to obtain in equation solely in terms of the energy density W, a relationship between the energy flow and the energy density $P = \mathfrak{J}(W)$ has to be exhibited. The general form of the $\mathfrak{J}$-operator is difficult to obtain. Indeed, this operator mainly depends on the propagative behaviour of the system studied. In the literature, the thermal analogy is often used to describe the relationship between active energy flow and total energy density [8–10].

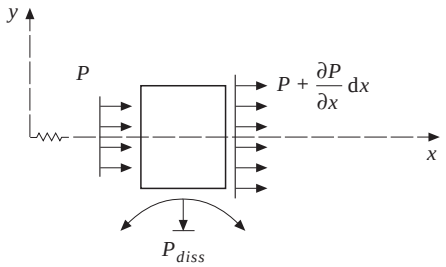

Figure 1. The energy flow balance in an elementary section.

Furthermore, for straight rods, Euler–Bernoulli beams and plane waves in membranes or plates, it has often been verified that the $\mathfrak{J}$-operator is a pure gradient one. Wohlever *et al*. [11, 12] provided an explicit relationship from the kinematic model of vibrating systems. Lase *et al*. [13–17] showed this to arise in a simplification of the general energy method, GEM. Hence, it is agreed in the literature that the conduction analogy is the model which best represents the flow of mechanicl energy. However, such a generalization has to be established rigorously.

In this study, interest is in the exhibition of the energy $\mathfrak{J}$-operator for several classes of systems, allowing, a review of the thermal analogy domain of validity. For this purpose, other energy quantities called partial energy densities and partial energy flows are introduced. Such quantities are defined as the active energy flow and the total energy density associated with each propagative wave. As just waveguides are studied here, two directions of propagation are possible: the x-positive one and the x-negative one. The corresponding partial energy quantities will be denoted by a superscript, as $W^{\pm}$ and $P^{\pm}$. An optional subscript is added in cases, of multi-propagative modes. The x-positive quantities are sometimes called incident quantities and the x-negative ones, reflected quantities.

Assumption (v) ensures that the global energies are the linear superposition of partial ones. This important principle can be interpreted as stemming from vanishing expectation of interference. Thus, assuming a random one-dimensional wave field composed of several uncorrelated propagative waves a_i, one finds that the expectation of the cross product of two waves

$$\mathrm{E}[a_i a_j^*] = a_i a_j^* \delta_{ij}, \tag{3}$$

where the asterisk denotes the complex conjugate and δ_{ij} is the Kronecker symbol. Furthermore, the global energy quantities are proportional to the square modulus of the sum of waves $\Sigma_i a_i$, so that cross-products $a_i a_j^*$ appear. The energy quantities W and P of the energy models are simply the expectation of the previous ones W, $P \propto \mathrm{E}[\Sigma_{i,j} a_i a_j^*]$. It is then clear that equation (3) implies that the global energy quantities W and P are the sums of the partial energy quantities $W^{\pm}$ and $P^{\pm}$. A similar assumption has been used by Langley [6] for multi-dimensional cases in order to derive a Wave Intensity Analysis, an SEA extension allowing a direct study of energy within systems.

Finally, it should be noted that evanescent waves are not taken into account in the previous calculation. This is exactly the purpose of the hypothesis (iv). Indeed, the near field vanishes rapidly far from singularities and especially as the frequency increases.

3. SEM FOR ONE SYMMETRIC PROPAGATIVE MODE SYSTEMS

In this section, one-dimensional systems with a unique symmetric propagative mode are studied. With a view to description by energy quantities, the behaviour of such systems is characterized by no more than two group velocities. Energy is transported by both an incident and a reflected wave, the group velocities of which are simply opposed in this case. One-dimensional systems which have this property are very numerous. One can cite strings, rods and Euler–Bernoulli beams, all well studied in previous works [11]. In this section, the wave approach is introduced and, in agreement with previous studies, Fourier's law is derived. As an illustration of the strategy developed, the particular case of a straight beam on an elastic support is studied explicitly.

3.1. Construction of the simplified energy model

The group velocities of the one-dimensional systems considered are denoted by c_g^+ and c_g^-. According to the symmetry of the propagation, one has $c_g^+ = -c_g^- = c_g$. In the case of an undamped system, the incident (respectively reflected) energy flow and the incident (respectively reflected) energy density are simply proportional [19]. The proportionality factor is the group velocity:

$$P^{\pm} = c_g^{\pm} W^{\pm}. \tag{4}$$

The relationship (4) may also be considered as a definition of the speed of propagation of energy. Note that despite the fact that the damping ratio is not taken into account in relationship (4), this constitutive law is considered to be valid, and the damping effect is assumed to be very slight (iii). The damping effect is then introduced by way of the dissipative term in the energy balance. On the other hand, the partial energy balances take the form

$$\mathrm{d}P^{\pm}/\mathrm{d}x + \eta\omega W^{\pm} = 0 \tag{5}$$

Combining relationships (4) and (5) yields:

$$\mathrm{d}(c_g^{\pm} W^{\pm})/\mathrm{d}x + \eta\omega P^{\pm}/c_g^{\pm} = 0, \tag{6}$$

or

$$P^{\pm} = -(c_g^2/\eta\omega)\,\mathrm{d}W^{\pm}/\mathrm{d}x. \tag{7}$$

As previously discussed, a linear superposition principle is valid for energy quantities and the global energy quantities are merely the sums of the partial ones,

$$W = W^+ + W^-, \qquad P = P^+ + P^-, \tag{8}$$

and hence, from relationships (7) and (8) one obtains the following relationship between the global variables:

$$P = -(c_g^2/\eta\omega)\mathrm{d}W/\mathrm{d}x. \tag{9}$$

The $\mathfrak{J}$-operator introduced in section 1 thus is a simple gradient operator for systems with one symmetric propagating mode. This law appears widely in the high frequency literature [8–17]. These texts often compare it with Fourier's law of heat conduction. According to this analogy, the flow of mechanical energy diffuses in structures as does the flow of thermal energy in a heat conduction problem.

By substituting relationship (9) into the energy balance (1), a diffusion equation similar to the heat conduction equation in steady state conditions, with a convective term, is obtained:

$$-(c_g^2/\eta\omega)\,\mathrm{d}^2W/\mathrm{d}x^2 + \eta\omega W = 0. \tag{10}$$

The solution of such an equation is the superposition of incident and reflected energies, the magnitudes of which must be calculated from a set of energy boundary conditions and coupling conditions. The problem of determining those energy conditions for longitudinally vibrating rods and transversely vibrating beams has been commented on in depth. The results given in reference [20] are a general form of the energy boundary construction and can be applied directly in the case studied here. For instance, the injected power P_{inj} is assumed to be known at the source. At a conservative end (clamped end, free end, simply supported end, etc.) the energy flow is equal to zero, $P = 0$ (see Figure 2). This is the well known power flow method developed by Nefske and Sung [10] and investigated

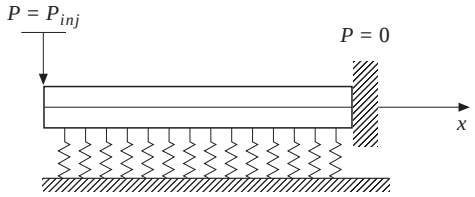

Figure 2. The boundary energy conditions for one symmetric propagative systems.

further by Wohlever and Bernhard [11], Lase *et al*. [13] and many others. A formulation such as this has been discussed at length, and is numerically attractive for mid- and high-frequency dynamics.

The problems treated in the literature are longitudinally vibrating rods and transversely vibrating Euler–Bernoulli beams, but equation (10) is quite valid for all systems with one symmetric propagative mode, as proved above. Hence, vibrating strings, transversely vibrating beams on elastic foundations, prestressed vibrating beams, etc., can be correctly described by equation (10). In the next section, a transversely vibrating beam on an elastic support is considered, as an example.

3.2. The governing displacement equation and energies for the beam on elastic support

Consider a transversely vibrating Euler–Bernoulli beam on an elastic foundation (see Figure 3(a)). A particular element of the beam is shown in Figure 3(b). For the sake of simplicity, it is assumed that the foundation is undamped. Let K denote the stiffness value of the foundation, ρ the mass per unit length, E the complex modulus of the material and I the inertia. Moreover, let v denote the transverse displacement, $\vartheta = \partial v/\partial x$ the rotation,

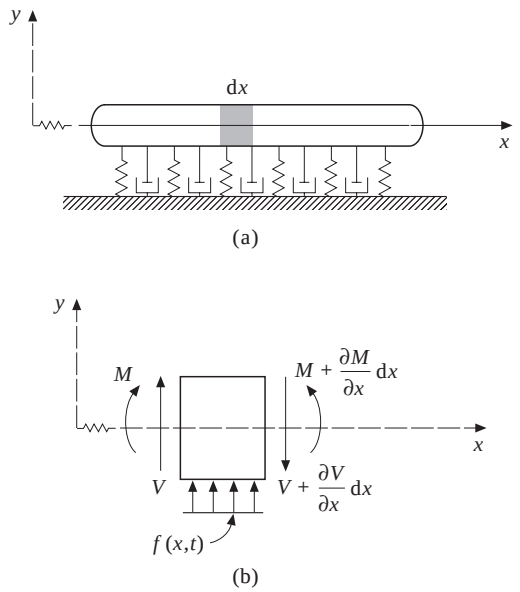

Figure 3. (a) A beam under tension on a visco-elastic foundation and (b) an elementary section of that beam.

$M = -EI\,\partial^2 v/\partial x^2$ the moment curvatures and $V = EI\,\partial^3 v/\partial x^3$ the shear force of the beam alone. The governing equation for such a system [21] free of excitation is

$$EI(\partial^4 v/\partial x^4) + Kv + \rho(\partial^2 v/\partial t^2) = 0. \tag{11}$$

The propagation behaviour of this system is described by considering that the displacement v is a harmonic wave proportional to $e^{i(\gamma x - \omega t)}$. The dispersion equation is obtained by introducing this form of wave in the governing equation (11). Hence

$$\gamma^4 - (k_0^4 - K/EI) = 0, \tag{12}$$

where $k_0^4 = \rho\omega^2/EI$ designates the simple Euler–Bernoulli wavenumber corresponding to $K = 0$. This frequency dispersion law has an interesting behaviour. Indeed, the natures and values of the number γ depend on the sign of the second part of equation (12). The solutions γ^+ and γ^- of this equation are

$$\gamma^{\pm} = \pm(k_0^4 - K/EI)^{1/4},\ \pm i(k_0^4 - K/EI)^{1/4}, \qquad \omega > \omega_c, \tag{13}$$

with $\omega_c^2 = K/\rho$. If the frequency is greater than the cut-off frequency, $\omega > \omega_c$, solutions (13) show two evanescent waves and two propagative waves which are simply opposed. By contrast, if $\omega < \omega_c$, the solutions of equation (12) are all evanescent waves [21]. This special case is not under consideration in this study: the frequency of excitation is assumed to be large enough above the cut-off frequency.

The group velocity associated with a wavenumber γ is given by $c_g = d\omega/d\gamma$. By using the propagative solutions of equation (13), γ^+ and γ^-, the group velocities can be readily obtained as

$$c_g^+ = (2\omega/\gamma^+)(1 - K/\rho\omega^2), \qquad c_g^- = (2\omega/\gamma^-)(1 - K/\rho\omega^2), \tag{14}$$

with $\gamma^+ = -\gamma^-$. From these expressions, it can be observed that the transversely vibrating beam on an elastic support has the required symmetry for $c_g^- = -c_g^+$. As shown in Figure 4, for $\omega > \omega_c$ the system has a single symmetric propagative mode. The simplified energy model of such a system is then the one defined in the previous section.

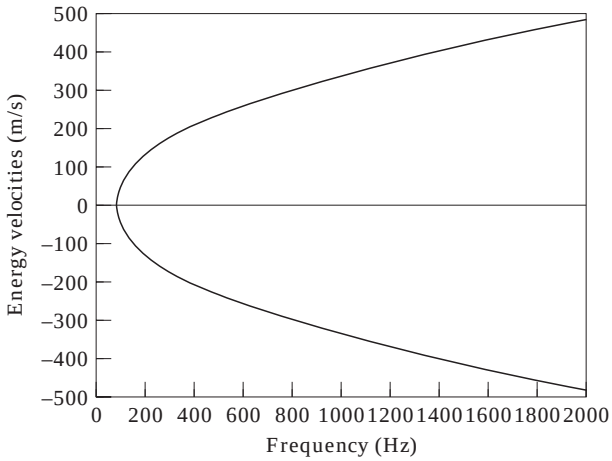

Figure 4. The frequency evolution of the group velocities for beam on elastic support.

One can now turn to the energy variables. For this system and with a harmonic time dependence, they take the following forms:

$$W = \tfrac{1}{4}\rho\omega^2|v|^2 + \tfrac{1}{4}E_0 I\left|\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right|^2 + \tfrac{1}{4}K|v|^2, \qquad P = \tfrac{1}{2}\mathrm{Re}\left(\mathrm{i}EI\omega\left(\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}v^* - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\frac{\mathrm{d}v^*}{\mathrm{d}x}\right)\right). \quad (15)$$

The energy density W is the sum of a kinetic energy density which has the same expression as for Euler–Bernoulli beams and a potential energy density which is simply the sum of the deformation energy density increased by the stiffness foundation effect. It can also be noticed that the energy flow is produced by the shear force and the bending moment, and that its expression is exactly the one employed in the motion of beams without a foundation. The partial energy variables used in the previous section can be calculated by calling v^+ and v^- the pure propagative waves of the system. Their expressions are respectively $v^+ \propto \mathrm{e}^{\mathrm{i}\gamma^+ x}$ and $v^- \propto \mathrm{e}^{\mathrm{i}\gamma^- x}$. By substituting the pure propagative waves into relationships (15), the partial incident and reflected energy density and energy flow are found to be

$$W^\pm = \tfrac{1}{4}(\rho s\omega^2 + E_0 I|\gamma^\pm|^4 + K)|v^\pm|^2, \qquad P^\pm = \tfrac{1}{2}\mathrm{Re}(\mathrm{i}\omega EI(-\mathrm{i}\gamma^{\pm 3} - \mathrm{i}\gamma^{\pm 2}\gamma^{\pm *}))|v^\pm|^2. \quad (16)$$

At this stage, several remarks can be made about the undamped case. First, without damping, the wavenumbers $\gamma^\pm$ have purely real values, and, as shown in expression (13), $\rho\omega^2 = E_0 I\gamma^4 + K$. Then, upon referring to equation (16), the partial kinetic energy density is seen to be equal to the partial potential energy density. Second, the required proportionality between the partial energy density and the partial energy flow given in equation (4) can be shown from the relationship (16). The proportionality factors (i.e., energy and velocity) are exactly the group velocities given by equations (14).

Finally, all of the assumptions used in the previous section are satisfied for the structure studied. The flow of mechanical energy of the transversely vibrating beam on elastic support is correctly modelled by equation (10).

In Figure 5 is shown the spatial evolution of energy density and energy flow for the free-clamped beam on an elastic support illustrated in Figure 3. The beam considered here is harmonically excited by a point force $F_{ex} = 1\mathrm{N}$. The characteristics of the beam are as follows: $\rho S = 7{\cdot}8$ kg/m, $E_0 I = 166{\cdot}66$ Nm^2, length $L = 2$ m. The distributed stiffness K is taken to be equal to 2×10^6 N/m^2. The damping loss factor is $\eta = 5 \pm 10^{-2}$. The frequency of excitation is $f = 1000$ Hz. The cut-off frequency of the beam is then $f_c = 80$ Hz. It should be noted that $f \geqslant f_c$.

At a frequency $f = 1000$ Hz the wavenumber is $\gamma = 30{\cdot}8$ m^{-1} and the beam contains almost 12 wavelengths, as shown in Figure 5(a). It is also shown in this figure that the simplified energy model provides the spatial average of the energy density predicted from the equation of motion (11) by using the energy expressions (15). The simplified energy flow evolution given in Figure 5(b) correctly smooths the "exact" one. The energy flow method developed here is the correct power flow model for one-dimensional systems with a unique symmetric mode of propagation. The equations are analogous to those for heat conduction and as the diffusion equation is numerically cheaper to solve than the wave equation, this is attractive for mid- and high-frequency dynamics. A small list of configurations that satisfy the assumptions is shown in Table 1.

In the following sections, more complicated systems are considered: namely, vibrating systems with a non-symmetric propagative mode and systems with multiple propagative modes.

4. SEM FOR NON-SYMMETRIC PROPAGATIVE MODE SYSTEMS

The one-dimensional structures studied in this section have a non-symmetric propagative mode. The propagative characteristics of such systems are provided by two different group velocities instead of the two opposite group velocities of the systems considered in section 3. The simplified energy model for such systems is derived and compared to the classical model. As an illustration of the method, a simple case of free–clamped pipes containing fluid is treated.

4.1. Construction of the simplified energy model

The energy velocities introduced in section 3 and denoted by c_g^+ and c_g^- are not assumed to be opposite. The relationships given in equation (4) for the associated undamped system are still valid. Moreover, the energy balances (5) for partial quantities remain valid also. Multiplying each of equations (5) by the non-corresponding energy velocity $c_g^{\pm}$ and summing them yields:

$$(\mathrm{d}/\mathrm{d}x)(c_g^+ I^- + c_g^- I^+) + \eta\omega(c_g^- W^+ + c_g^+ W^-) = 0, \tag{17}$$

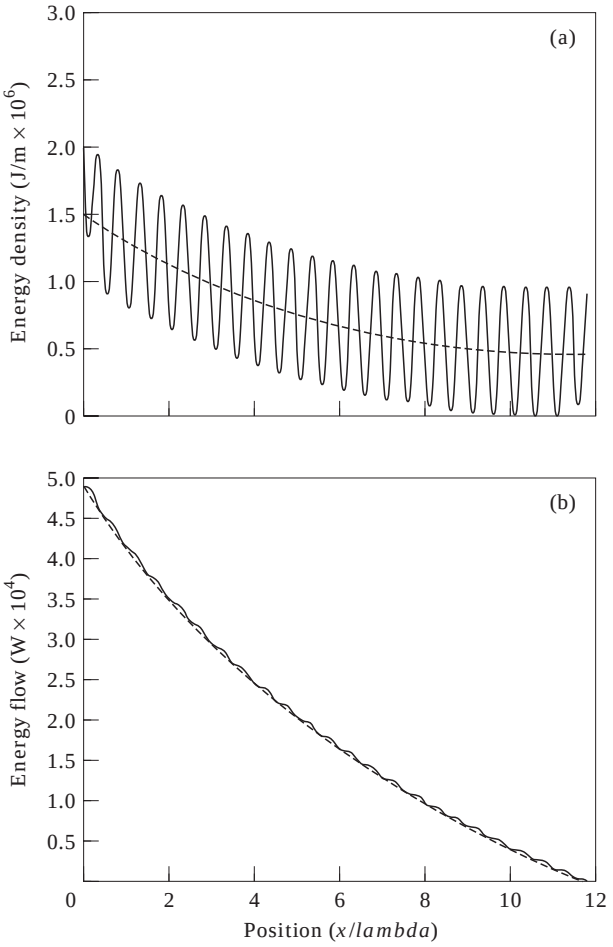

Figure 5. The spatial evolution of (a) the total energy density and (b) the active energy flow for (——) the kinematic model and (---) for the SEM solutions for the beam on pure elastic support.

TABLE 1

Some vibrating systems of the class of systems with one propagative mode: total energy density, active energy flow and real group velocity are given

Configurations	Energy density	Energy flow	Real group velocity
Strings	$\frac{\rho\omega^2}{4}\lvert u\rvert^2+\frac{T_0}{4}\left\lvert\frac{\mathrm{d}u}{\mathrm{d}x}\right\rvert^2$	$-\frac{\mathrm{i}\omega}{2}T\frac{\mathrm{d}u}{\mathrm{d}x}u^*$	$c_g=\frac{\omega}{k_0}$ $k_0=\omega\sqrt{\rho/T_0}$
Rods	$\frac{\rho S\omega^2}{4}\lvert u\rvert^2+\frac{E_0S}{4}\left\lvert\frac{\mathrm{d}u}{\mathrm{d}x}\right\rvert^2$	$-\frac{\mathrm{i}\omega}{2}ES\frac{\mathrm{d}u}{\mathrm{d}x}u^*$	$c_g=\frac{\omega}{k_0}$ $k_0=\omega\sqrt{\rho/E_0}$
Euler beams	$\frac{\rho S\omega^2}{4}\lvert v\rvert^2+\frac{E_0I}{4}\left\lvert\frac{\mathrm{d}v}{\mathrm{d}x}\right\rvert^2$	$-\frac{\mathrm{i}\omega}{2}EI\left(\frac{\mathrm{d}^3v}{\mathrm{d}x^3}v^*-\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\frac{\mathrm{d}v^*}{\mathrm{d}x}\right)$	$c_g=\frac{2\omega}{k_0}$ $k_0=(\rho S\omega^2/E_0I)^{1/4}$

or

$$(\mathrm{d}/\mathrm{d}x)(c_g^+ I^- + c_g^- I^+) + \eta\omega(c_g^- + c_g^+)(W^+ + W^-) = \eta\omega(c_g^+ W^+ + c_g^- W^-). \tag{18}$$

Then, by applying the relationships (4), the global energy flow in terms of energy density is found to be

$$P = (c_g^+ c_g^- / \eta\omega)(\mathrm{d}W/\mathrm{d}x) + \underbrace{(c_g^+ + c_g^-)W}_{\text{new term}}. \tag{19}$$

The required $\mathfrak{J}$-operator which defines the mechanical energy flow behaviour of this particular system is different from the one defined in equation (9) of section 3. Indeed, the $\mathfrak{J}$-operator of equation (19) is not a pure gradient one, but is the superposition of a gradient of energy density and a new term which is simply proportional to the energy density. It appears clearly that the heat conduction analogy previously used is not valid for system with non-symmetric propagative modes. By taking $c_g^+ = -c_g^-$ in equation (19), however the relationship (9) is recovered, so equation (19) is a generalization of the Fourier's law (9).

The equation for the energy density is derived by substituting equation (19) in the energy balance (1), and is

$$(c_g^+ c_g^- / \eta\omega)\mathrm{d}^2W/\mathrm{d}x^2 + (c_g^+ + c_g^-)\mathrm{d}W/\mathrm{d}x + \eta\omega W = 0. \tag{20}$$

Hence, the energy equation exhibited here is a generalized form of the heat energy formulation analogy (10) shown in section 3. Note that the energy model (20) is still a second order differential equation and thus is still numerically efficient. The solutions of equations (20) and (19) are simply

$$W = A\mathrm{e}^{(\eta\omega/c_g^+)x} + B\mathrm{e}^{-(\eta\omega/c_g^-)x}, \qquad P = c_g^+ A\mathrm{e}^{-(\eta\omega/c_g^+)x} + c_g^- B\mathrm{e}^{-(\eta\omega/c_g^-)x}, \tag{21}$$

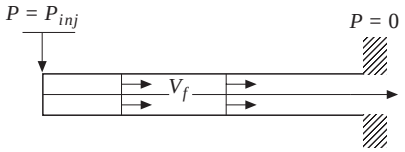

Figure 6. The boundary energy conditions for non-symmetric propagative systems.

where the constants A and B denote the energy magnitudes. To determine these parameters the boundary and coupling conditions proposed in section 3 can be applied here (see Figure 6). The results of the method developed in this section can be illustrated by a simple case of fluid-carrying pipes. The loss of symmetry for such a problem is due to the velocity of the fluid.

4.2. The governing displacement equation and energies for fluid-filled pipes

The system under consideration (see Figure 7(a)), consists of a tube [22, 23] of mass per unit length m_b, elasticity modulus E and inertia I. This tube is conveying fluid of mass par unit length m_f, which flows with a constant velocity V_f. The analysis presented here is carried out for the transverse vibrational displacement v of the fluid–structure system.

The governing equation for the system is [23, 24].

$$EI(\partial^4 v/\partial x^4) + m_f V_f^2(\partial^2 v/\partial x^2) + 2m_f V_f(\partial^2 v/\partial t \partial x) + (m_f + m_b)\partial^2 v/\partial t^2 = 0. \tag{22}$$

In this equation the effect of the pressure on the transverse displacement of the tube wall is neglected. Obviously, the equation obtained by setting the fluid mass per unit length equal to zero is the simple Euler–Bernoulli beam equation of motion. At this stage, the propagative wave behaviour needs to be examined. For a pure propagative wave proportional to $e^{i(\gamma x - \omega t)}$, the resulting dispersion equation is

$$\gamma^4 - (m_f V_f^2/EI)\gamma^2 + (2\omega m_f V_f/EI)\gamma - \omega^2(m_f + m_b)/EI = 0. \tag{23}$$

The solutions of this equation are easier to check numerically than analytically. However, it can be readily verified that the roots of equation (23) are not sign opposed, as the given dispersion equation is not of a bi-squared form. The tube conveying fluid chosen here satisfies the non-symmetry property, and hence its energy model is the one given above. The numerical approach highlights that just two solutions of equation (23) have propagative behavior; the two others correspond to evanescent waves.

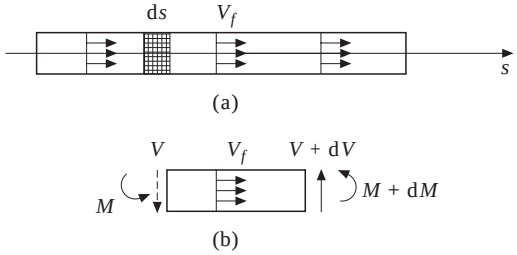

Figure 7. (a) A curved tube conveying fluid and (b) an elementary section of that tube.

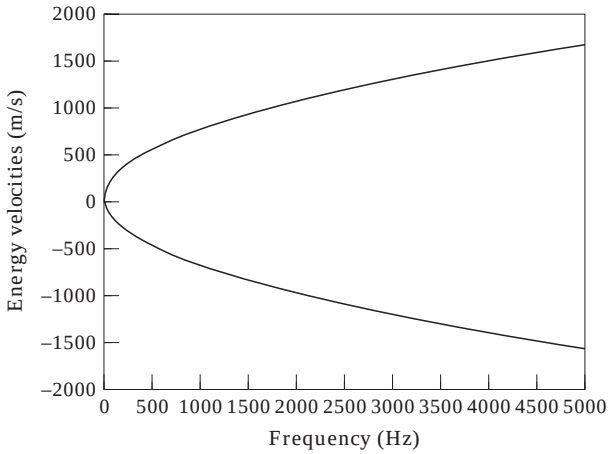

Figure 8. The frequency evolutions of the energy veolcities for tube conveying fluid.

The total energy density and the active energy flow for such a system, for harmonic time dependence are given by

$$W = (m_b\omega^2/4)|v|^2 + (m_f/4)|V_f \mathrm{d}v/\mathrm{d}x - \mathrm{i}\omega v|^2 + (E_0 I/4)|\mathrm{d}^2 v/\mathrm{d}x^2|^2,$$

$$P = \tfrac{1}{2}\mathrm{Re}\left(\mathrm{i}\omega m_f V_f \left(V_f \frac{dv}{dx} v^* - \mathrm{i}\omega |v|^2\right] + \mathrm{i}\omega EI \left(\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} v^* - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\frac{\mathrm{d}v^*}{\mathrm{d}x}\right)\right). \tag{24}$$

The partial energy densities and energy flows are deduced from expressions (24) in a similar way as those for systems with a symmetric propagative mode. Then, substituting $v^{\pm} \propto \mathrm{e}^{\mathrm{i}\gamma^{\pm}x}$ into relationships (24) yields

$$W^{\pm} = \tfrac{1}{4}(m_b\omega^2 + m_f|\gamma^{\pm}V_f - \omega|^2 + E_0 I|\gamma^{\pm}|^4)|v^{\pm}|^2,$$

$$P^{\pm} = \tfrac{1}{2}\mathrm{Re}(\mathrm{i}\omega m_f V_f(\mathrm{i}\gamma^{\pm}V_f - \mathrm{i}\omega) + \mathrm{i}\omega EI(-\mathrm{i}\gamma^{\pm^3} - \mathrm{i}\gamma^{\pm^2}\gamma^{\pm^*}))|v^{\pm}|^2. \tag{25}$$

It is now obvious that the partial energy flow is proportional to the corresponding partial energy density. By definition, the proportionality factor is the energy velocity. A plot of the energy velocities versus frequency is shown in Figure 8.

Moreover, it can readily be demonstrated that the kinetic energy density is equal to the potential energy density. In the undamped case, equation (23) can be factorized as $EI\gamma^4 = m_f(V_f\gamma - \omega)^2 + m_b\omega^2$, which is exactly the condition required by equation (25) for this equality.

Finally, for a numerical simulation, the characteristics of the tube are taken as follows: length $L = 2$ m, solid mass per unit length $m_b = 4{\cdot}68$ kg/m, fluid mass per unit length $m_f = m_b$. The damping loss factor is $\eta = 5 \times 10^{-2}$. The fluid characteristics are chosen in order to simulate significant high frequency dynamics, and to avoid the pipe's limit of stability. In fact, the fluid velocity is taken to be equal to $V_f = V_c/1{\cdot}1$, where $V_c = 45{\cdot}9\ \mathrm{ms}^{-1}$ is the critical speed, $V_c = (\pi/L)\sqrt{E_0 I/m_f}$. The beam is excited by a transverse force $F_{ex} = 1$ N. The excitation frequency is taken to be equal to $f = 3000$ Hz. At this frequency, the energy velocities are $c_g^+ = 1206\ \mathrm{ms}^{-1}$ and $c_g^- = -1289\ \mathrm{ms}^{-1}$. Comparison between results obtained from the kinematic analysis by solving the equation of motion (22) and a set of boundary conditions written in terms of displacement and force, and the energetic calculation are given in Figures 9(a) and 9(b). It is shown that the evolution of the energy variables calculated from equations (22) and (25) by using the

kinematic model has a very oscillating aspect due to interference. While introduction of the energy model eliminates the oscillations from the response. In general this permits one to use a reasonable mesh of the structure for numerical calculations rather than the finer one needed to solve the governing equation (22).

5. SEM FOR MULTI-MODE OF PROPAGATION SYSTEMS

The examples treated in previous sections illustrate the strategy adopted for an energetic description and its efficiency for analysis of the mid- and high-frequency dynamics. However, these examples are representative only of systems with one propagative mode. More realistically, one-dimensional systems contain numerous propagative modes, which cannot always be neglected. In this section the problem of systems with two distinct symmetric propagative modes is considered. Note that the strategy developed can be generalized to more complicated situations with more propagative modes. The results here are applied to Timoshenko beams.

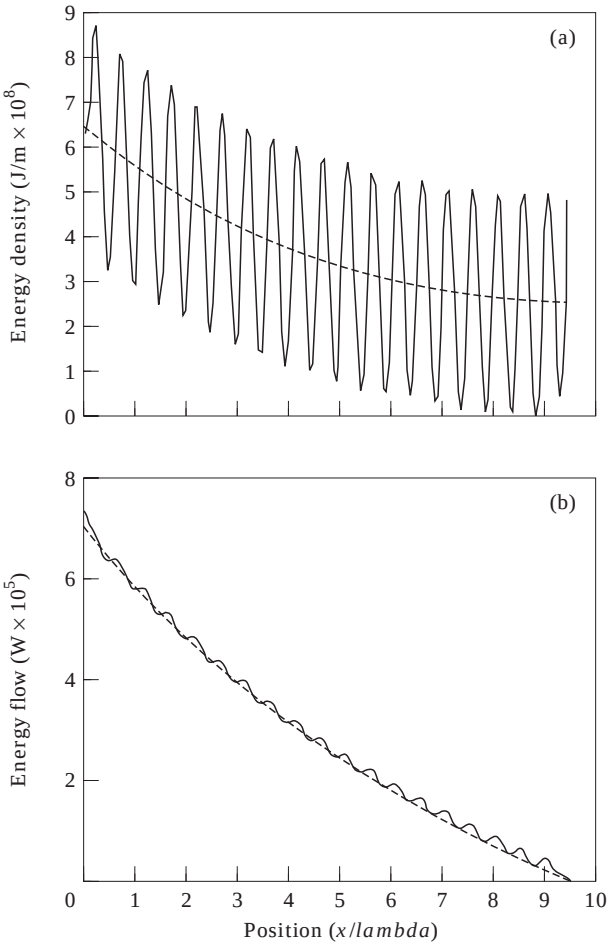

Figure 9. The spatial evolution of (a) the total energy density and (b) the active energy flow for (——) the kinematic model and (---) for the SEM solutions for the straight beam conveying fluid.

5.1. Construction of the smooth energy model

The systems taken into consideration in this section are characterized by four energy velocities, denoted by c_{g1}^{+}, c_{g1}^{-} and c_{g2}^{+}, c_{g2}^{-}. The subscript number refers to the number of the mode of propagation. The assumed symmetry implies that the energy velocities are opposed for a given mode of propagation. Therefore $c_{g1}^{+} = -c_{g1}^{-} = c_{g1}$ and $c_{g2}^{+} = -c_{g2}^{-} = c_{g2}$. The relationships between the partial energy flows and the corresponding partial energy densities are exactly as in sections 3 and 4:

$$P_i^{\pm} = c_{gi}^{\pm} W_i^{\pm}, \qquad i = 1,2. \tag{26}$$

Moreover, four partial energy balances, given in the damped case, have to be written:

$$\mathrm{d}P_i^{\pm}/\mathrm{d}x + \eta\omega W_i^{\pm} = 0, \qquad i = 1,2. \tag{27}$$

The global energy quantities associated with a given mode of propagation, numbered i, are defined as

$$W_i = W_i^{+} + W_i^{-}, \quad P_i = P_i^{+} + P_i^{-}, \qquad i = 1,2. \tag{28}$$

Note that the physical global energy variables W and P are, respectively, the sums of the new partial energy densities W_1, W_2 and P_1, P_2: $P = P_1 - P_2$ and $W = W_1 + W_2$. By considering expressions (26–28), it can be seen that they constitute two uncoupled systems. Each of them is similar to the problem of equations (4), (5) and (8) treated in section 3. As a consequence, equations (9) and (10) still apply, so

$$P_i = -(c_{gi}^2/\eta\omega)\,\mathrm{d}W_i/\mathrm{d}x, \quad -(c_{gi}^2/\eta\omega)\,\mathrm{d}^2W_i/\mathrm{d}x^2 + \eta\omega W_i = 0, \qquad i = 1,2. \tag{29, 30}$$

Then, two uncoupled differential problems are obtained for the energy quantities W_i and P_i. The relationships (29) can be involved as boundary conditions. These differential problems may possibly be coupled because of these boundary conditions.

Another presentation is possible in terms of the physical quantities W and P. Taking a linear combination of equations (29) yields

$$c_{g1}c_{g2}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(W_1 + W_2) + \eta\omega\left(\frac{c_{g2}}{c_{g1}}\frac{\mathrm{d}^2P_1}{\mathrm{d}x^2} + \frac{c_{g1}}{c_{g2}}\frac{\mathrm{d}^2P_2}{\mathrm{d}x^2}\right) = 0. \tag{31}$$

Now, by substituting equations (27) into equation (31) and rearranging the result, one obtains

$$c_{g1}c_{g2}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(W_1 + W_2) - (\eta\omega)^2\left(\frac{c_{g2}}{c_{g1}} + \frac{c_{g1}}{c_{g2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}(W_1 + W_2) = -(\eta\omega)^2\left(\frac{c_{g2}}{c_{g1}}\frac{\mathrm{d}W_2}{\mathrm{d}x} + \frac{c_{g1}}{c_{g2}}\frac{\mathrm{d}W_1}{\mathrm{d}x}\right). \tag{32}$$

By combining the relationships (29) obtained above and using the global energy quantities W and P, the required intrinsic $\mathfrak{J}$-operator can be obtained as in previous sections:

$$P = \frac{(c_{g1}c_{g2})^2}{\eta^3\omega^3}\frac{\mathrm{d}^3W}{\mathrm{d}x^3} - \left(\frac{c_{g1}^2}{\eta\omega} + \frac{c_{g2}^2}{\eta\omega}\right)\frac{\mathrm{d}W}{\mathrm{d}x}. \tag{33}$$

The $\mathfrak{J}$-operator here is seen to be a third order derivative operator. In addition, once again it can be noted that the previous relationship is a generalization of the heat conduction analogy obtained in section 3. Upon neglecting a kind of propagation in the system, for instance the second propagative mode, the corresponding group velocity vanishes, and substituting this result in equation (33) recovers the well-known Fourier law (9).

A fourth order energy equation in terms of the global energy density W can be written as

$$\frac{(c_{g1}c_{g2})^2}{\eta^3\omega^3}\frac{\mathrm{d}^4W}{\mathrm{d}x^4} - \left(\frac{c_{g1}^2}{\eta\omega} + \frac{c_{g2}^2}{\eta\omega}\right)\frac{\mathrm{d}W}{\mathrm{d}x^2} + \eta\omega W = 0. \tag{34}$$

The solutions of equations (34) and (33) are

$$\begin{aligned} W &= A_1\mathrm{e}^{-(\eta\omega/c_{g1}^+)x} + B_1\mathrm{e}^{-(\eta\omega/c_{g1}^-)x} + A_2\mathrm{e}^{-(\eta\omega/c_{g2}^+)x} + B_2\mathrm{e}^{-(\eta\omega/c_{g2}^-)x},\\ P &= c_{g1}^+A_1\mathrm{e}^{-(\eta\omega/c_{g1}^+)x} + c_{g1}^-B_1\mathrm{e}^{-(\eta\omega/c_{g1}^-)x} + c_{g2}^+A_2\mathrm{e}^{-(\eta\omega/c_{g2}^+)x} + c_{g2}^-B_2\mathrm{e}^{-(\eta\omega/c_{g2}^-)x}, \end{aligned} \tag{35}$$

Where constants A_i and B_i are the energy magnitudes. They are determined by some appropriate boundary conditions. For instance, two injected powers are assumed to be known at the excitation point. At a conservative end, the energy flows are equal to zero (see Fig. 10).

It thus appears that the derivative order of the simplified energy equation of a system depends mainly on its number of propagative branches. The strategy elaborated here shows that an energy model can be given explicitly from the knowledge of the energy velocities. In the next section, an example of such a one-dimensional system is considered: namely, a Timoshenko beam which includes both effects of shear and rotary inertia.

5.2. The governing displacement equations and energies for Timoshenko beams

A differential element of a straight beam subjected to a shear force V and a bending moment M is considered, as shown in Figure 11(a). The displacement of the centroidal axis is denoted by V and the slope of the centroidal axis is given by $\vartheta = \partial v/\partial x$. A new co-ordinate ψ to measure the slope of the cross-section due to bending. The beam under consideration has a complex modulus E, a transverse modulus G, an inertia I, a section s and a mass per unit volume ρ. κ is the adjustment coefficient which depends on the shape of the cross-section. The essential features of the shear deformation development are shown in Figure 11(b).

The governing equations for Timoshenko beams are [20]:

$$\begin{aligned} Gs\kappa(\partial\psi/\partial x - \partial^2 v/\partial x^2) + \rho s(\partial^2 v/\partial t^2) = 0,\\ Gs\kappa(\partial v/\partial x - \psi) + EI(\partial^2\psi/\partial x^2) - \rho I(\partial^2\psi/\partial t^2) = 0. \end{aligned} \tag{36}$$

As mentioned in reference [20], there are two modes of deformation in this theory. One mode of deformation represents simply the transverse deflection v of the beam. The other mode represents the transverse shearing deformation measured by $\partial v/\partial x - \psi$. The Timoshenko beam is an example of a multi-propagative mode system.

The dispersion equation of the Timoshenko beam can now be determined. Two approaches are possible. The set of coupled equations of motion (36) may be used directly or may be reduced to a single equation. The more direct method is used here. A harmonic

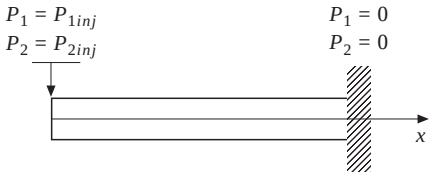

Figure 10. The boundary energy conditions for two symmetric propagative systems.

wave admits the form $v = a\mathrm{e}^{\mathrm{i}(\gamma x - \omega t)}$. Substituting these expressions into the governing equations (36) yields

$$(Gs\kappa\gamma^2 - \rho s\omega^2)a + \mathrm{i}Gs\kappa\gamma b = 0, \qquad \mathrm{i}Gs\kappa\gamma a - (Gs\kappa + EI\gamma^2 - \rho I\omega^2)b = 0. \tag{37}$$

The dispersion equation of the Timoshenko beam is then obtained by considering that such a harmonic wave can exist only if the determinant of the system (37) vanishes. Hence

$$(Gs\kappa\gamma^2 - \rho s\omega^2)(Gs\kappa + EI\gamma^2 - \rho I\omega^2) - (Gs\kappa\gamma)^2 = 0 \qquad \text{or} \qquad \gamma^4 - \tau\gamma^2\chi = 0, \tag{38}$$

where $\tau = (\rho\omega^2/E)(1 + E/G\kappa)$ and $\chi = (\rho\omega^2/E)((\rho\omega^2/G\kappa) - s/I)$. The coefficients a and b are then proportional to each other: with $\mathrm{i}\alpha$ denoting the constant of proportionality one has $\psi = \mathrm{i}\alpha v$.

It should be noted that the dispersion characteristics of Timoshenko beam are close to those of exact theory of three-dimensional elasticity, in contrast with the Euler–Bernoulli model. It was pointed out in reference [20] that the results provided by the Timoshenko model for thin beams are in agreement with exact theory, especially at high frequencies.

The main information needed for the energy equation is the number and the nature of the roots of the dispersion equation. In this case, as this equation is simply bi-square, the required symmetry (roots two by two, sign opposed) is established. Expression (38) shows also that the parameter τ, which is positive, is the sum of the squares of the two roots. Therefore, the nature of the roots depends on the sign of the parameter χ, the product of the squares of the roots.

A transition frequency (cut-off frequency) exists. This frequency is calculated by assigning χ to be equal to zero. This expression is simply

$$\omega_c^2 = \kappa Gs/\rho I. \tag{39}$$

It corresponds to an increase of propagative branches from two symmetric modes to four modes that are two by two symmetric. Timoshenko beams have an attractive propagation behaviour, as there is a one symmetric propagative mode system for frequencies below the

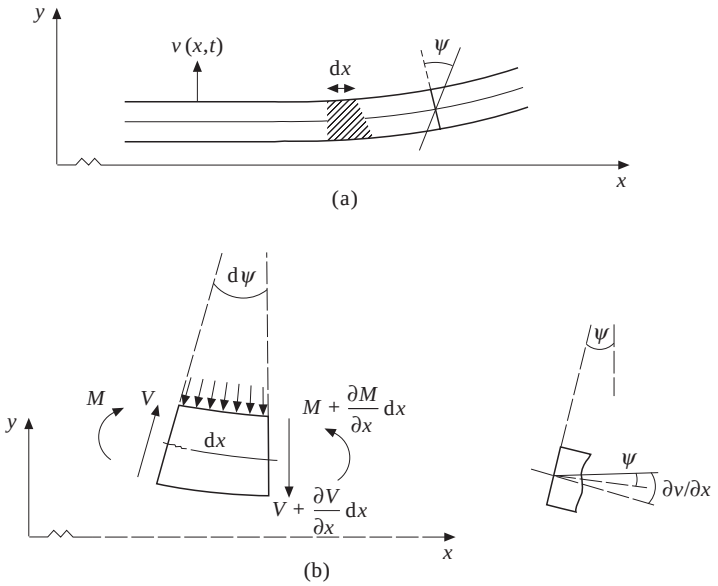

Figure 11. (a) A Timoshenko beam configuration, and (b) an elementary section.

cut-off frequency, with two symmetric propagative modes for frequencies above the cut-off frequency.

The total energy density and energy flow for the Timoshenko vibrating beam with a harmonic time dependence are as follows:

$$W = \frac{\rho s\omega^2}{4}|v|^2 + \frac{\rho I\omega^2}{4}|\psi|^2 + \frac{E_0 I}{4}\left|\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|^2 + \kappa\frac{G_0 s}{4}\left|\frac{\mathrm{d}v}{\mathrm{d}x} - \psi\right|^2,$$

$$P = \tfrac{1}{2}\mathrm{Re}\left(\mathrm{i}\omega EI\frac{\mathrm{d}\psi}{\mathrm{d}x}\psi^* + \mathrm{i}\omega\kappa Gs\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \psi\right)v^*\right). \tag{40}$$

The partial energy densities and energy flows are deduced from expressions (40) by substituting into them $v_i^{\pm} = a\mathrm{e}^{\mathrm{i}v_i^{\pm}x}$ and $\psi_i^{\pm} = \mathrm{i}\alpha a\mathrm{e}^{\mathrm{i}v_i^{\pm}x}$, which yields

$$W_i^{\pm} = \tfrac{1}{4}(\rho s\omega^2 + \rho I\omega^2|\alpha|^2 + E_0 I|\gamma\alpha|^2 + \kappa G_0 s|\gamma - \alpha|^2)|v_i^{\pm}|^2,$$

$$P_i^{\pm} = \tfrac{1}{2}\mathrm{Re}(-\omega EI\gamma|\alpha|^2 - \mathrm{i}\omega\kappa Gs(\gamma - \alpha)v^*)|v_i^{\pm}|^2. \tag{41}$$

As in previous sections, it can be seen that the partial energy flow is proportional to the corresponding partial energy density. The evolution of the energy velocities deduced from expressions (41) versus frequency is shown in Figure 12.

Moreover, after a few calculations on the first expression (41), it can be seen from the dispersion equation (38) that, in the undamped case, the kinetic energy density is equal to the potential energy density.

To provide a numerical example, the kinematic model is solved by using the classical governing equation (36), for the case of a transverse shear force imposed at the origin. These kinematic results are compared to those of the simplified energy model solution of equation (35). The Timoshenko beam considered here is harmonically excited by a point force $F_{ex} = 1$ N. The characteristics of the beam are as follows: $\rho S = 1320$ kg/m, $E_0 I = 625\,000$ Nm^2, $\kappa G_0 s = 1153$ N, length $L = 2$ m. The radius of gyration is equal to $\sqrt{I/s} = 0{\cdot}143$ m; the damping loss factor is $\eta = 10^{-2}$; the frequency of excitation is $f = 200$ Hz; the cut-off frequency of the beam is $f_c = 103{\cdot}09$ Hz.

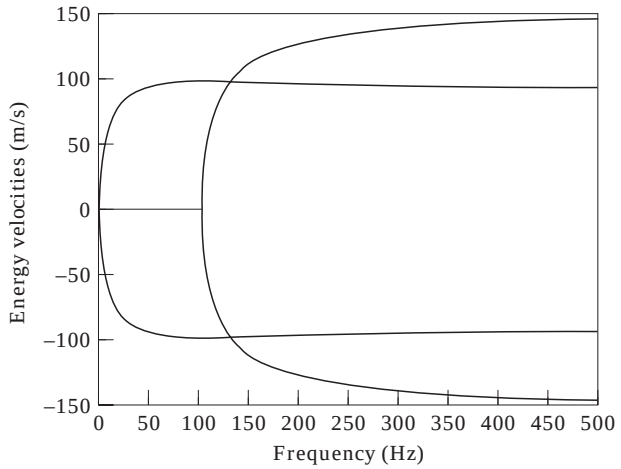

Figure 12. The frequency evolutions of the energy velocities for Timoshenko beam.

In Figures 13(a) and 13(b) are given the energy density and the energy flow evolution versus the x- abscissa in the Timoshenko beam. Both the classical calculation and the simplified energy calculation are given. From the comparative energy results given in Figure 13, the existence of a multi-mode of propagation is highlighted. Indeed, the classical energy solutions contain two forms of propagation corresponding to the two wavelengths. The simplified energy predictions contain just the macroscopic behavior of the former. Finally, the simplified energy method is an efficient tool for the mid- and high-frequency representation of vibrating systems.

6. CONCLUDING REMARKS

Some energy flow properties have been studied in depth in this paper. Three classes of one-dimensional systems have been reviewed to introduce three possible schemes of energy flow representation. It is shown that the propagative law of systems which contain only two symmetric propagative modes is the simple heat conduction model. In this case, the space average concept used by other authors and assumption (v) are equivalent. Further

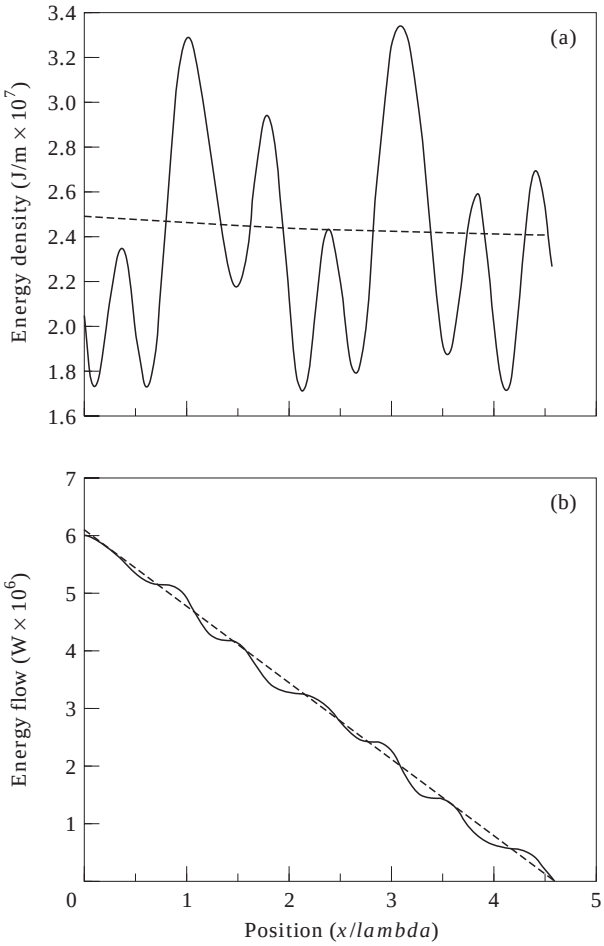

Figure 13. The spatial evolution of (a) the total energy density and (b) the active energy flow for (——) the kinematic model and (---) for the SEM solutions for the Timoshenko beam.

new propagative laws have been exhibited for systems with non-symmetric propagative modes and for systems with multi-propagative behaviour. A similar process can be employed to deal with energy flow models of wave guides with more than four propagative group velocities. Moreover, this technique can be used systematically to study the medium and high frequency dynamics of wave guides from solely knowledge of the dispersion curves. More realistic models well suited for increasing frequency than the usually Euler–Bernoulli beam can be considered. For instance, one-dimensional systems with useful sections, (I-section, U-section, etc.) can be represented.

ACKNOWLEDGMENTS

The authors gratefully acknowledge Mr R. Aquilina (Centre d'Etude et Recherche de la Discrétion Acoustique des Navires) and Mr Luzzato (Direction des Études et Recherches EDF) for their kind advice, suggestions and support.

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